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# THE EQUILIBRIUM OF AN ELASTIC SPACE WEAKENED BY TWO SPHERICAL CAVITIES AND AN EXTERNAL CIRCULAR CRACK†

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Using the relationship between the basic solutions of Laplace's equation in toroidal and spherical coordinates, the Fourier method is employed to solve the problem of the equilibrium of an elastic space weakened by two spherical cavities and an external circular crack. The proposed approach leads to an infinite system of linear algebraic equations of the second kind with exponentially decaying matrix coefficients. A small-parameter expansion is used to obtain an asymptotic formula for the normal stress intensity factor.

1. LET $\alpha, \beta, \varphi; \alpha, \sigma, \varphi; r, \theta, \varphi;\ r_1, \theta_1, \varphi_1, \rho, z, \varphi; \rho_1, z_1, \varphi_1$ be toroidal, spherical, and cylindrical coordinates defined by the following formulae [1–3]

$$x = ah_\beta^{-2}\operatorname{sh}\alpha\cos\varphi,\quad y = ah_\beta^{-2}\operatorname{sh}\alpha\sin\varphi,\quad z = ah_\beta^{-2}\sin\beta$$

$$x = ah_\sigma^{-2}\operatorname{sh}\alpha\cos\varphi,\quad y = ah_\sigma^{-2}\operatorname{sh}\alpha\sin\varphi,\quad z = ah_\sigma^{-2}\sin\sigma$$

$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\quad z = r\cos\theta$$

$$x = \rho\cos\varphi,\quad y = \rho\sin\varphi,\quad z = z$$

$$x_1 = x,\quad y_1 = y,\quad z_1 = z - h$$

$$\rho_1 = \rho,\quad z_1 = z - h;\quad \rho_1 = r_1\sin\theta_1,\quad z_1 = r_1\cos\theta_1$$

$$(a > 0,\ h \geqslant 0;\ 0 \leqslant \alpha, \rho, \rho_1, r, r_1 < \infty;\ -\infty < z, z_1 < \infty,\ -\pi \leqslant \beta, \sigma \leqslant \pi,$$

$$0 \leqslant \varphi \leqslant 2\pi,\quad 0 \leqslant \theta, \theta_1 \leqslant \pi,$$

$$h_\beta = \sqrt{\operatorname{ch}\alpha + \cos\beta},\quad h_\sigma = \sqrt{\operatorname{ch}\alpha - \cos\sigma}$$

In the case of a homogeneous isotropic elastic body the equilibrium equations can be reduced to the Lamé vector equation

$$\frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u} + \Delta\mathbf{u} = 0 \tag{1.1}$$

Here $\mathbf{u}$ is the elastic displacement vector and $\nu$ is Poisson's ratio.

The relationships between the basic solutions of (1.1) in spherical and toroidal coordinates can be obtained from the following equalities relating the basic solutions of Laplace's equation in these coordinates (the factor $e^{im\varphi}$ is omitted on both sides of each equality)

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$$\left(\frac{a}{r_1}\right)^{n+1} P_n^m(\cos\theta_1) = h_\sigma \int_{-\infty}^{\infty} b_n^{(m)}(\tau) e^{\tau\sigma} P^m_{-1/2+i\tau}(\operatorname{ch}\alpha)d\tau \tag{1.2}$$

$$\left(-\pi - 2\operatorname{arctg}\frac{h}{a} < \sigma < \pi - 2\operatorname{arctg}\frac{h}{a}\right)$$

$$h_\beta e^{-\tau\beta} P^m_{-1/2+i\tau}(\operatorname{ch}\alpha) = \sum_{n=0}^{\infty} c_n^{(m)}(\tau)\left(\frac{r_1}{a}\right)^{n+m} P^m_{n+m}(\cos\theta_1) \tag{1.3}$$

$$(r_1 < \sqrt{a^2+h^2})$$

$$\left(\frac{r_1}{a}\right)^n P_n^m(\cos\theta_1) = h_\beta \int_{-\infty}^{\infty} a_n^{(m)}(\tau) e^{\tau\beta} P^m_{-1/2+i\tau}(\operatorname{ch}\alpha)d\tau (|\beta| < \pi) \tag{1.4}$$

$$h_\sigma e^{-\tau\sigma} P^m_{-1/2+i\tau}(\operatorname{ch}\alpha) = \sum_{n=0}^{\infty} d_n^{(m)}(\tau)\left(\frac{a}{r_1}\right)^{n+m+1} P^m_{n+m}(\cos\theta_1) \tag{1.5}$$

$$(r_1 > \sqrt{a^2+h^2})$$

Here

$$a_n^{(m)}(\tau) = (-i)^{n-m}\frac{2^{m-1/2}(n+m)!}{(2m)!(n-m)!}(1-i\varepsilon)^{n-m}\frac{F(m-n, 1/2 - i\tau + m; 2m+1; \gamma)}{\operatorname{ch}\pi\tau}$$

$$b_n^{(m)}(\tau) = \frac{a_n^{(m)}(\tau)}{(1+\varepsilon^2)^{n+1/2}} e^{2\tau\operatorname{arctg}\varepsilon}, \quad c_n^{(m)}(\tau) = \frac{d_n^{(m)}(\tau)}{(1+\varepsilon^2)^{n+m+1/2}} e^{-2\tau\operatorname{arctg}\varepsilon},$$

$$d_n^{(m)}(\tau) = \frac{(-1)^m(-i)^n 2^{m+1/2}}{(2m)!}\frac{\Gamma(1/2 + i\tau + m)}{\Gamma(1/2 + i\tau - m)}(1-i\varepsilon)^n F(-n, 1/2 + i\tau + m; 2m+1; \gamma)$$

$$F(-n, a; c; z) = \sum_{m=0}^{n} \frac{(a)_m(-n)_m}{(c)_m m!} z^m, \quad \varepsilon = \frac{h}{a},$$

$$(\alpha)_m = \alpha(\alpha+1)\ldots(\alpha+m-1), \quad \gamma = \frac{2}{1-i\varepsilon}$$

$p_n^m(x)$ are associated Legendre polynomials, $P_\nu^m(z)$ are associated Legendre functions of the first kind, $\Gamma(z)$ is the gamma-function, $F(-n, a; c; z)$ is the hypergeometric polynomial in $z$, and $(\alpha)_m$ is the Pochhammer symbol [3, 4].

The method of obtaining expansions of the type (1.2)–(1.5) and using them to solve the scalar and vector boundary-value problems of elasticity theory is well known [5–7]. For $m=1$ the expansions (1.2)–(1.5) enable us to study a number of problems on twisting: (a) a body $\beta_1 \leqslant \beta \leqslant \beta_2$ weakened by a spherical cavity $0 \leqslant r_1 \leqslant R$ or several disjoint spherical cavities with centres on the $z$ axis; (b) a sphere $0 \leqslant r_1 \leqslant R$ with a cavity $\beta_1 \leqslant \beta \leqslant \beta_2$.

**2.** Let $\rho_2$, $z_2$, $\varphi_2$; $r_2$, $\theta_2$, $\varphi_2$ be the cylindrical and spherical coordinates defined by

$$\rho_2 = r_2\sin\theta_2, \quad z_2 = r_2\cos\theta_2; \quad \rho_2 = \rho_1, \quad z_2 = -z_1 - 2h = -z - h$$

We will consider the equilibrium problem for an elastic space weakened by two spherical cavities $0 \leqslant r_1 \leqslant R$ and $0 \leqslant r_2 \leqslant R$ symmetrical about the plane $z=0$ and an external circular crack (cut) $\beta = \pm\pi$ (see Fig. 1). We will confine ourselves to the case when the crack edges are free from any external forces and do not touch one another, while the surfaces of the cavities are

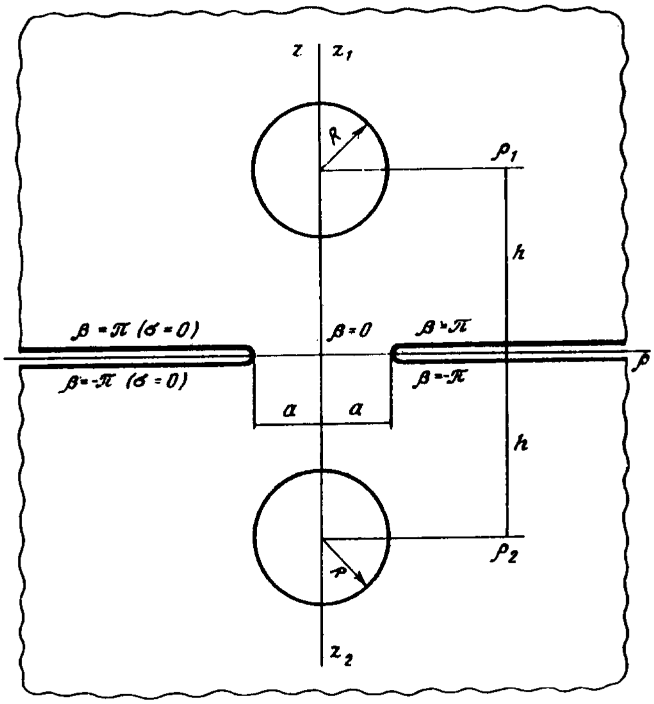


FIG. 1.

subject to a hydrostatic pressure of intensity $\sigma_0 > 0$.

The corresponding boundary conditions have the form

$$\sigma_{r1}\big|_{r_1=R} = \sigma_{r2}\big|_{r_2=R} = -\sigma_0 \tag{2.1}$$

$$\tau_{r\theta 1}\big|_{r_1=R} = \tau_{r\theta 2}\big|_{r_2=R} = 0; \quad \sigma_z\big|_{\beta=\pm\pi} = 0, \quad \tau_{\rho z}\big|_{\beta=\pm\pi} = 0 \tag{2.2}$$

($\sigma_{rj}$, $\tau_{r\theta j}$, $\sigma_z$, $\tau_{\rho z}$ are the components of the stress tensor in spherical and cylindrical coordinates).

We will represent the general solution of the vector equation (1.1) in the form

$$\mathbf{u} = (\kappa\mathbf{e}_z - z\,\mathrm{grad})\Phi + (\kappa\mathbf{e}_{z1} - z_1\,\mathrm{grad})F_1 + (\kappa\mathbf{e}_{z2} - z_2\,\mathrm{grad})F_2 - \mathrm{grad}(\varphi + f + \psi) \tag{2.3}$$

$$\Phi = h_\beta \int_0^\infty A(\tau)P_{-\frac{1}{2}+i\tau}(\operatorname{ch}\alpha)\operatorname{sh}\tau\beta\,d\tau, \qquad \frac{\partial\varphi}{fz} = (1-2\nu)\Phi$$

$$F_1 = \sum_{n=1}^\infty (2n-1)B_n^{(1)}r_1^{-n}P_{n-1}(\cos\theta_1), \quad F_2 = \sum_{n=1}^\infty (2n-1)B_n^{(1)}r_2^{-n}P_{n-1}(\cos\theta_2)$$

$$f = -\sum_{n=1}^\infty (n+3-4\nu)B_n^{(1)}[r_1^{-(n-1)}P_{n-2}(\cos\theta_1) + r_2^{-(n-1)}P_{n-2}(\cos\theta_2)]$$

$$\psi = -\sum_{n=0}^\infty B_n^{(2)}[r_1^{-n-1}P_n(\cos\theta_1) + r_2^{-n-1}P_n(\cos\theta_2)]$$

($z_1 = z-h$, $z_2 = -z-h$; $\kappa = 3-4\nu$, $\mathbf{e}_{z1}$, $\mathbf{e}_{z2}$, $\mathbf{e}_z$ are the unit vectors of the corresponding systems of coordinates), which ensures that (2.2) is an identity.

On verifying boundary conditions (2.1) using the decomposition formulae

$$r_1^{-n}P_{n-1}(\cos\theta_1)=a^{-n}h_\sigma\int_{-\infty}^{\infty}b_{n-1}^{(0)}(\tau)e^{\tau\sigma}P_{-1/2+i\tau}(\operatorname{ch}\alpha)d\tau$$

$$h_\beta P_{-1/2+i\tau}(\operatorname{ch}\alpha)\operatorname{sh}\tau\beta=-\sum_{n=0}^{\infty}\bar c_n(\tau)\,a^n\,r_1^{-n}P_n(\cos\theta_1)$$

$$r_2^{-n-1}P_n(\cos\theta_2)=(2h)^{-n-1}\sum_{k=0}^{\infty}(-1)^{n+k}\frac{(n+k)!}{n!k!}(2h)^{-k}r_1^kP_k(\cos\theta_1)\ (r_1<2h)$$

$$\bar c_n(\tau)=\tfrac{1}{2}[c_n^{(0)}(\tau)-c_n^{(0)}(-\tau)]$$

which follow from (1.2) and (1.3) and the equality [8]

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^kP_k(x)=w^{-a/2}F\left(a,1-a;1;\tfrac{1}{2}-\frac{1-tx}{2\sqrt{w}}\right)\qquad (w=1-2tx+t^2)$$

for $a=n+1$, $t=-r_1/(2h)$ and $x=\cos\theta_1$, after some reduction we find the following system of relationships between the integral density $A(\tau)$ and the coefficients $B_n^{(1)}$ and $B_n^{(2)}$

$$B_1^{(1)}=0$$

$$A(\tau)=-\sum_{n=2}^{\infty}B_n^{(1)}a^{-n}(n^2-2n-1+2\nu)\frac{\bar b_{n-1}(\tau)}{\operatorname{ch}\pi\tau}-\sum_{n=2}^{\infty}n(2n-1)B_n^{(1)}ha^{-n-1}\frac{\bar b_n(\tau)}{\operatorname{ch}\pi\tau}-$$
$$-\sum_{n=0}^{\infty}(n+1)B_n^{(2)}a^{-n-2}\frac{\bar b_{n+1}(\tau)}{\operatorname{ch}\pi\tau}$$

$$B_0^{(2)}R^{-3}=\frac{1+\nu}{3}\sum_{n=2}^{\infty}(-1)^nn(2n-1)(2h)^{-n-1}B_n^{(1)}+\frac{1+\nu}{3}a^{-1}q_1-\frac{\sigma_0}{4G}$$

$$-B_k^{(1)}R^{-k}k(k^2+3k-2\nu)+B_k^{(2)}R^{-k-2}(k+1)(k+2)=$$
$$=-\sum_{n=2}^{\infty}B_n^{(1)}(-1)^{n+k}\frac{(n+k)!}{(n-1)!k!}\frac{2n-1}{2k+3}(k^2-k-2-2\nu)\frac{R^{k-1}}{(2h)^{n+k-1}}-$$
$$-\sum_{n=2}^{\infty}B_n^{(1)}(-1)^{n+k-1}\frac{(n+k-1)!(k-1)}{(n-1)!(k-1)!}\frac{2nk-n-k+4-4\nu}{2k-1}\frac{R^{k+1}}{(2h)^{n+k+1}}-$$
$$-\sum_{n=0}^{\infty}B_n^{(2)}(-1)^{n+k}\frac{(n+k)!(k-1)}{n!(k-1)!}\frac{R^{k-1}}{(2h)^{n+k+1}}-k(k-1)\frac{h}{a}\left(\frac{R}{a}\right)^{k-1}q_k-$$
$$-\frac{k+1}{2k+3}(k^2-k-2-2\nu)\left(\frac{R}{a}\right)^{k+1}q_{k+1}-\frac{k-1}{2k-1}(k^2-2k-1+2\nu)\left(\frac{R}{a}\right)^{k-1}q_{k-1}$$

$$B_k^{(1)}R^{-k}k(k^2-2+2\nu)-B_k^{(2)}R^{-k-2}k(k+2)=$$
$$=-\sum_{n=2}^{\infty}B_n^{(1)}(-1)^{n+k}\frac{(n+k)!k}{(n-1)!(k+1)!}\frac{2n-1}{2k+3}(k^2+2k-1+2\nu)\frac{R^{k+1}}{(2h)^{n+k+1}}-$$
$$-\sum_{n=2}^{\infty}B_n^{(1)}(-1)^{n+k-1}\frac{(n+k-1)!(k-1)}{(n-1)!(k-1)!}\frac{2nk-n-k+4-4\nu}{2k-1}\frac{R^{k-1}}{(2h)^{n+k-1}}-$$
$$-\sum_{n=0}^{\infty}B_n^{(2)}(-1)^{n+k}\frac{(n+k)!(k-1)}{n!(k-1)!}\frac{R^{k-1}}{(2h)^{n+k+1}}-k(k-1)\frac{h}{a}\left(\frac{R}{a}\right)^{k-1}q_k-$$

$$-\frac{k}{2k+3}(k^2+2k-1+2\nu)\left(\frac{R}{a}\right)^{k+1} q_{k+1}-\frac{k-1}{2k-1}(k^2-2k-1+2\nu)\left(\frac{R}{a}\right)^{k-1} q_{k-1}$$

$(k=1,2,\ldots)$

$$q_s=\int_0^\infty A(\tau)\bar{c}_s(\tau)d\tau,\quad \bar{b}_s(\tau)=2[b_s^{(0)}(\tau)-b_s^{(0)}(-\tau)]$$

Now, eliminating $A(\tau)$ and setting

$$\frac{\sigma_0}{4G}\frac{1}{k}R^{k+1}b_k^{(1)}=B_k^{(1)}\quad (k=2,3\ldots),\qquad \frac{\sigma_0}{4G}\frac{2k-1}{2k+1}R^{k+3}b_k^{(2)}=B_k^{(2)}\quad (k=0,1,2,\ldots)$$

$$\omega_k=\frac{2(k^2+2k\nu+k+1+\nu)}{(k+2)(2k-1)(2k+3)},\qquad \beta_{nk}^{(0)}=\frac{(-1)^{n+k+1}(n+k)!(2n-1)}{2^{n+k+1}n!(k+1)!}$$

$$\Delta_k=k^2-2k\nu+k+1-\nu,\quad \alpha_{nk}^{(0)}=\frac{(-1)^{n+k}(n+k)!}{2^{n+k+2}\Delta_k n!(k-1)!},\quad \alpha_k^{(1)}=\frac{k(k^2-1)}{2\Delta_k}$$

$$\alpha_k^{(2)}=\frac{k(k-1)(2k+1)}{2\Delta_k},\quad \alpha_k^{(3)}=\frac{(k-1)(2k+1)}{2\Delta_k(2k-1)}(k^2-2k-1+2\nu)$$

$$\beta_{nk}^{(1)}=(2n-1)(k-1),\quad \beta_{nk}^{(2)}=\frac{2n-1}{2n+1}(2k+1)(k-1)$$

$$\gamma_{nk}^{(1)}=\frac{4(2k+1)(k-1)}{(n+k)(2k-1)}(2nk-n-k+4-4\nu),\quad \gamma_n=\frac{n^2-2n-1+2\nu}{n}$$

$$\delta_n=\frac{(n+1)(2n-1)}{2n+1},\quad \lambda=\frac{R}{h},\quad \varepsilon=\frac{h}{a}$$

we obtain the following infinite system of linear algebraic equations from which to determine $b_k^{(1)}$ and $b_k^{(2)}$

$$b_k^{(1)}=\sum_{n=2}^\infty D_{kn}^{(11)}b_n^{(1)}+\sum_{n=0}^\infty D_{kn}^{(12)}b_n^{(2)}\quad (k=2,3,\ldots) \tag{2.4}$$

$$b_k^{(2)}=\sum_{n=2}^\infty D_{kn}^{(21)}b_n^{(1)}+\sum_{n=0}^\infty D_{kn}^{(22)}b_n^{(2)}+f_k^{(2)}\quad (k=0,1,2,\ldots)$$

$$f_0^{(2)}=1,\quad f_k^{(2)}=0\quad (k=1,2,\ldots)$$

$$D_{0n}^{(21)}=\frac{1+\nu}{3}[(-1)^{n+1}(2n-1)2^{-n-1}+\gamma_n\varepsilon^{n+1}J_{n-11}+(2n-1)\varepsilon^{n+2}J_{n1}]\lambda^{n+1} \tag{2.5}$$

$$D_{0n}^{(22)}=\frac{1+\nu}{3}\delta_n\varepsilon^{n+3}J_{n+11}\lambda^{n+3}$$

$$D_{kn}^{(11)}=\{\alpha_{nk}^{(0)}\beta_{nk}^{(1)}-\alpha_k^{(1)}\varepsilon^{n+k+1}[\gamma_nJ_{n-1k+1}+(2n-1)\varepsilon J_{nk+1}]\}\lambda^{n+k+1}-$$
$$-\{\alpha_{nk}^{(0)}\gamma_{nk}^{(1)}+\alpha_k^{(2)}\varepsilon^{n+k}[\gamma_nJ_{n-1k}+(2n-1)\varepsilon J_{nk}]+$$
$$+\alpha_k^{(3)}\varepsilon^{n+k-1}[\gamma_kJ_{n-1k-1}+(2n-1)\varepsilon J_{nk-1}]\}\lambda^{n+k-1}$$

$$D_{kn}^{(12)}=(\alpha_{nk}^{(0)}\beta_{nk}^{(2)}-\alpha_k^{(2)}\varepsilon^{n+k+2}\delta_nJ_{n+1k}-\alpha_k^{(3)}\varepsilon^{n+k+1}\delta_nJ_{n+1k-1})\lambda^{n+k+1}-$$
$$-\alpha_k^{(1)}\varepsilon^{n+k+3}\delta_nJ_{n+1k+1}\lambda^{n+k+3}$$

$$D_{kn}^{(21)}=D_{kn}^{(11)}-[\beta_{nk}^{(0)}+\gamma_n\varepsilon^{n+k+1}J_{n-1k+1}+(2n-1)\varepsilon^{n+k+2}J_{nk+1}]\omega_k\lambda^{n+k+1}$$

$$D_{kn}^{(22)}=D_{kn}^{(12)}-\omega_k\delta_n\varepsilon^{n+k+3}J_{n+1k+1}\lambda^{n+k+3}$$

$$J_{mk}=\int_0^\infty\frac{\bar{b}_m(\tau)\bar{c}_k(\tau)}{\operatorname{ch}\pi\tau}d\tau=\tfrac{1}{2}J_{mk}^{(1)}-\tfrac{1}{2}J_{mk}^{(2)}$$

$$J_{mk}^{(1)} = \int_{-\infty}^{\infty} \frac{c_m^{(0)}(-\tau)c_k^{(0)}(\tau)}{\mathrm{ch}^2 \pi\tau} d\tau, \qquad J_{mk}^{(2)} = \int_{-\infty}^{\infty} \frac{c_m^{(0)}(\tau)c_k^{(0)}(\tau)}{\mathrm{ch}^2 \pi\tau} d\tau$$

Taking into account that [4, 9]

$$F(-n;\tfrac{1}{2}-i\tau;1;\gamma) = \sum_{s=0}^{n} \frac{(-n)_s \Gamma(\tfrac{1}{2}-i\tau+s)}{(s!)^2 \Gamma(\tfrac{1}{2}-i\tau)} \gamma^s, \quad \frac{\pi}{\mathrm{ch}\,\pi\tau} = \Gamma(\tfrac{1}{2}+i\tau)\Gamma(\tfrac{1}{2}-i\tau)$$

$$\int_{-\infty}^{\infty} \Gamma(\tfrac{1}{2}+i\tau)\Gamma(\tfrac{1}{2}-i\tau)\Gamma(\tfrac{1}{2}-i\tau+j)\Gamma(\tfrac{1}{2}+i\tau+l)d\tau = 2\pi \frac{j!\,l!}{j+l+1}$$

we can rewrite $J_{mk}^{(1)}$ in the form

$$J_{mk}^{(1)} = \frac{4}{\pi}(-i)^{m+k} \frac{(1-i\varepsilon)^{m+k}}{(1+\varepsilon^2)^{m+k+1}} \Sigma_{mk}$$

$$\Sigma_{mk} = \sum_{j=0}^{m} \frac{(-m)_j \gamma^j}{j!} \sum_{l=0}^{k} \frac{(-k)_l \gamma^l}{l!(j+l+1)}, \quad \gamma = \frac{2}{1-i\varepsilon}$$

Since

$$\sum_{s=0}^{n} \frac{(-n)_s (\gamma x)^s}{s!} = (1-\gamma x)^n$$

it follows that

$$\Sigma_{mk} = \int_0^1 (1-\gamma x)^{m+k} dx = \frac{1-(1-\gamma)^{m+k+1}}{\gamma(m+k+1)}$$

which implies that

$$J_{mk}^{(1)} = \frac{2}{\pi} i(-1)^{m+k+1} \frac{(\varepsilon+i)^{m+k+1} - (\varepsilon-i)^{m+k+1}}{(m+k+1)(1+\varepsilon^2)^{m+k+1}} = $$
$$= \frac{4}{\pi}(-1)^{m+k} \sum_{j=0}^{m+k+1} C_{m+k+1}^j \varepsilon^j \sin\frac{\pi(m+k+1-j)}{2} \frac{1}{(m+k+1)(1+\varepsilon^2)^{m+k+1}} \quad (2.6)$$

For $J_{mk}^{(2)}$, whose structure is more complex, it is only possible to obtain the recurrent relations

$$J_{mk+1}^{(2)} = -\frac{1}{2\varepsilon}[J_{mk}^{(2)} + J_{m-1k+1}^{(2)}] \quad (m = 1,2,\ldots; \quad k = 0,1,2,\ldots)$$
$$J_{0k+1}^{(2)} = -\frac{1}{2\varepsilon} J_{0k}^{(2)} + \frac{2}{\pi} \frac{(-1)^k}{\varepsilon(k+1)(1+\varepsilon^2)^{k+1}} \sum_{j=0}^{k+1} C_{k+1}^j \varepsilon^j \sin\frac{\pi(k+1-j)}{2} \quad (2.7)$$
$$J_{00}^{(2)} = \frac{4}{\pi}\frac{\mathrm{arctg}\,\varepsilon}{\varepsilon}; \quad J_{km}^{(2)} = J_{mk}^{(2)} \quad (m,k = 0,1,2,\ldots)$$

Formulae (2.6) and (2.7) are suitable for computing the matrix coefficients (2.5) of the infinite system (2.4), but they are of little use in studying the properties of that system. A preliminary analysis of the matrix coefficients indicates that to study the properties of the infinite system (2.4) for various relations between $a$, $R$, and $h$, it suffices to investigate the behaviour of $J_{mk}^{(2)}(\varepsilon)$ as $m+k \to \infty$. To estimate $J_{mk}^{(2)(\varepsilon)}$ we use the Cauchy–Bunyakovski inequality

$$[J_{mk}^{(2)}(\varepsilon)]^2 \leqslant J_{mm}^{(2)}(\varepsilon) J_{kk}^{(2)}(\varepsilon) \quad (J_{nn}^{(2)}(\varepsilon) \geqslant 0)$$

Representing $J_{nn}^{(2)}(\varepsilon)$ in the form

$$J_{nn}^{(2)}(\varepsilon) = \int_{-\infty}^{\infty} \frac{e^{4\tau \operatorname{arctg} \varepsilon}}{\operatorname{ch}^2 \pi\tau} q_n^2(\tau) d\tau, \quad q_n^2(\tau) > 0$$

$$q_n(\tau) = (-i)^n \sqrt{2} \frac{(1-i\varepsilon)^n}{(1+\varepsilon^2)^{n+1/2}} F(-n, 1/2 - i\tau; 1; \gamma), \quad \gamma = \frac{2}{1-i\varepsilon}$$

and taking into account that $e^{2\tau \operatorname{arctg} \varepsilon} \leqslant 2 \operatorname{ch} \pi\tau$ $(0 < \varepsilon < \infty,\ -\infty < \tau < \infty)$, we have the inequality

$$J_{nn}^{(2)}(\varepsilon) \leqslant 4(-1)^n \frac{(1-i\varepsilon)^{2n}}{(1+\varepsilon^2)^{2n+1}} R_n(\varepsilon)$$

$$R_m(\varepsilon) = \int_{-\infty}^{\infty} \frac{e^{2\tau \operatorname{arctg} \varepsilon}}{\operatorname{ch} \pi\tau} F^2(-m, 1/2 - i\tau; 1; \gamma) d\tau = \frac{1}{\pi} \sum_{j=0}^{m} \frac{(-m)_j \gamma^j}{(j!)^2} \sum_{l=0}^{m} \frac{(-m)_l \gamma^l}{(l!)^2} T_{jl}$$

$$T_{jl} = \int_{-\infty}^{\infty} e^{2\tau \operatorname{arctg} \varepsilon} \frac{\Gamma(1/2 + i\tau)}{\Gamma(1/2 - i\tau)} \Gamma(1/2 - i\tau + j)\Gamma(1/2 - i\tau + l) d\tau =$$

$$= 2\pi e^{i \operatorname{arctg} \varepsilon} j! l! F(j+1, l+1; 1; -e^{2i \operatorname{arctg} \varepsilon})$$

Now, using the relations [8]

$$\sum_{l=0}^{\infty} \frac{(a)_l (b')_l}{l!(c')_l} t^l F(a+l, b; c; x) = F_2(a, b, b'; c; c'; x; t)$$

for $a=1,\ c'=1,\ c=1,\ b=j+1,\ b'=-m,\ t=\gamma,\ x=-\exp(2i \operatorname{arctg} \varepsilon)$ and

$$F_2(a, b, b'; a; a'; w; z) = (1-w)^{-b}(1-z)^{-b'} F\left(b, b'; a; \frac{wz}{(1-w)(1-z)}\right)$$

for $a=1,\ b=j+1,\ b'=-m,\ t=\gamma,\ w=-\exp(2i \operatorname{arctg} \varepsilon),\ z=\gamma$, after some reduction we arrive at the equality

$$R_n(\varepsilon) = (-1)^n \frac{(1+i\varepsilon)^n}{(1-i\varepsilon)^n} \sqrt{1+\varepsilon^2}$$

It follows that

$$[J_{mk}^{(2)}(\varepsilon)]^2 \leqslant 16(1+\varepsilon^2)^{-m-k-1} \quad (0 < \varepsilon < \infty) \tag{2.8}$$

Using (2.8) one can verify that for $0 < \lambda < 1$ and $i = 1, 2$

$$D_{kn}^{(i1)} \sim \alpha_{nk}^{(0)} \beta_{nk}^{(1)} \lambda^{n+k+1} = O(nk^{-1}(n+k) e^{-s(n+k+1)})$$

$$D_{kn}^{(i2)} \sim \alpha_{nk}^{(0)} \beta_{nk}^{(2)} \lambda^{n+k+1} = O((n+k) e^{-s(n+k+1)}) \quad (n+k \to \infty,\ s = -\ln \lambda)$$

i.e. the matrix elements of the infinite system (2.4) for $0 < \lambda < 1$ decay exponentially in each row and each column. Moreover, for $0 < \lambda = R/h < 1$

$$\sum_{k,n=1}^{\infty} [D_{kn}^{(ij)}]^2 < \infty, \quad \sum_{n=2}^{\infty} [D_{0n}^{(2j)}]^2 < \infty, \quad \sum_{k=2}^{\infty} [D_{k0}^{(i2)}]^2 < \infty \tag{2.9}$$

From (2.9) and the fact that $\{f_k^{(2)}\}$ belongs to the Hilbert space $l_2$ of number-valued sequences it follows that for almost all $\lambda \in (0, 1)$ a unique solution of the infinite system (2.4) in $l_2$ exists, which can be found by the reduction method [10, 11]. The estimates (2.9) enable us to conclude that the infinite system (2.4) is quasiregular for $0<\lambda<1$ and completely regular for $0<\lambda\leqslant\lambda_0<1$ for some $\lambda_0\in(0, 1)$.

The restriction $0<\lambda<1$ of possible values of $\lambda$ is connected in a natural way with the formulation of the problem in question and means that the spheres $r_1=R$ and $r_2=R$ do not intersect one another or have a point of contact.

Solving the infinite system (2.4) by the small-parameter method and confining ourselves to those terms that enable us to compute the normal stress intensity factor $K_1$ up to the terms of order $\lambda^6$ inclusive, we get

$$b_2^{(1)}=d_0\lambda^3+O(\lambda^4),\quad b_k^{(1)}=O(\lambda^{k+1})\quad (k=3,4\ldots) \tag{2.10}$$

$$b_0^{(2)}=1+\frac{1+\nu}{3}\delta_0\varepsilon^3 J_{11}\lambda^3+O(\lambda^4),\quad b_1^{(2)}=O(\lambda^4)$$

$$b_k^{(2)}=O(\lambda^{k+2})\quad (k=2,3,\ldots)$$

$$d_0=d_{02}^{(0)}\beta_{02}^{(0)}-\alpha_2^{(2)}\varepsilon^4\delta_0 J_{12}-\alpha_2^{(3)}\varepsilon^3\delta_0 J_{11}$$

$$J_{11}=\frac{1}{\pi}\left[-\frac{1}{\varepsilon^3}\operatorname{arctg}\varepsilon+\frac{1}{\varepsilon^2(1+\varepsilon^2)}-\frac{2}{3}\frac{1-3\varepsilon^2}{(1+\varepsilon^2)^2}\right]$$

$$J_{12}=\frac{1}{\pi}\left[\frac{3}{4\varepsilon^4}\operatorname{arctg}\varepsilon-\frac{1}{4\varepsilon^3(1+\varepsilon^2)}-\frac{1}{2\varepsilon(1+\varepsilon^2)^2}+\frac{2}{3}\frac{(1-3\varepsilon^2)}{(1+\varepsilon^2)^3}\right]$$

On the basis of the asymptotic solution (2.10), we have

$$K_1=\lim_{\rho\to a}[\sigma_z\sqrt{2(a-\rho)}]_{\beta=0}=-\frac{\sigma_0}{2}\sqrt{2a}\sum_{n=0}^{\infty}\{b_{n+2}^{(1)}[\gamma_{n+2}r_{n+1}+(2n+3)\varepsilon r_{n+2}]+$$

$$+b_n^{(2)}\delta_n r_{n+1}\}(\lambda\varepsilon)^{n+3}=\frac{2\sigma_0\sqrt{a}}{\pi}(1+\varepsilon^2)^{-2}\times$$

$$\times\left\{1-\varepsilon^2+\left[d_0(\varepsilon^*-\gamma_2+\gamma_2\varepsilon^2)-\frac{1+\nu}{3}\varepsilon^3(1-\varepsilon^2)J_{11}\right]\lambda^3\right\}\varepsilon^3\lambda^3+O(\lambda^7);$$

$$\varepsilon^*=\frac{3\varepsilon^2(3-\varepsilon^2)}{1+\varepsilon^2},\quad r_m=\frac{\sqrt{2}}{\pi}(-1)^m\frac{(\varepsilon+i)^{m+1}+(\varepsilon-i)^{m+1}}{(1+\varepsilon^2)^{m+1}}$$

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